by (II), (III) and (IV) and therefore tends to 0 as $i \to \infty$. Thus the matrix $\{c_{in}\}$ also satisfies (ii). But

$$\sum_{n=1}^{\infty} |c_{in}| = (i+1) \sum_{n=1}^{n_i} |b(m_i, n)| + i \sum_{n=1}^{\infty} |b(w_i, n)| \geqslant i \left| \sum_{n=1}^{\infty} b(w_i, n) \right| = i,$$

and hence $\{c_{in}\}$ does not satisfy ($c$) and is not a Toeplitz matrix.

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# THE FACTORIZATION OF LINEAR GRAPHS

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1. *Introduction.* We make use of the following definitions.

A *graph* is a finite simplicial 1-complex. The *order* of a graph is the number of its 0-simplexes. The *degree* of a 0-simplex in a graph is the number of 1-simplexes with which it is incident. If all the 0-simplexes in a graph $N$ have degree $\sigma$, $N$ is said to be *regular* and of the $\sigma$-th *degree.* A *component* of a graph is a connected part not contained in any larger connected part.

A *subgraph* of a graph $N$ is a graph consisting of all the 0-simplexes and some subset of the 1-simplexes of $N$. A *factor* is a regular subgraph of the first degree. If $N$ has no factor it is *prime*. Clearly all graphs of odd order are prime.

Let the 0-simplexes of a graph $N$ be enumerated as $a_1, a_2, \ldots, a_n$, and let $S = (a_i, a_j, \ldots, a_r)$ be any subset of them. Then we denote by $N_S$ or $N_{ij\ldots r}$ the graph obtained from $N$ by suppressing the 0-simplexes of $S$ and the 1-simplexes which are incident with members of $S$. We denote the number of members of $S$ by $f(S)$, the number of components of $N_S$ by $h(S)$ and the number of these components of odd order by $h_u(S)$. If $h(S) > 1$ we say that $S$ is an *isthmoid* of rank $f(S)$.

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In this paper, by a method based on the properties of skew-symmetric determinants, we show that a graph $N$ is prime if and only if it contains an $S$ such that $h_u(S) > f(S)$. We deduce from this that if $A$ is any 1-simplex of a connected regular graph of even order and of degree $\sigma$, having no isthmoid of rank $< \sigma - 1$, then $N$ has a factor which contains $A$. Those conversant with the theory of finite graphs will observe that this result contains Petersen's Theorem* as a special case.

2. *Pfaffians*. Let $\Delta = |c_{ij}|$ be a skew-symmetric determinant in which the elements above the diagonal are independent indeterminates over the ring of rational integers. It is known† that if $\Delta$ is of odd order it vanishes, but that if it is of even order $2m$ it is the square of a "Pfaffian". This Pfaffian $P$ is given by

$$P = \Sigma \, \epsilon \, c_{ij} \, c_{kl} \dots c_{rs} \tag{1}$$

in which the summation is over all partitions of the integers 1 to $2m$ into pairs $(i, j)$, $(k, l)$, ..., $(r, s)$, the order of the elements in each pair and the arrangement of the $m$ pairs being immaterial. $\epsilon$ is $+1$ or $-1$ according as the sequence $(i, j, k, l, \dots, r, s)$ is an even or odd permutation of $(1, 2, \dots, 2m)$. We assume hereafter that $\Delta$ is of even order $2m$.

Let $\Delta_{ij\dots r}$ be the skew-symmetric determinant obtained from $\Delta$ by striking out the $i$-th, $j$-th, ..., $r$-th rows and columns, and let $P_{ij\dots r}$ be the corresponding Pfaffian. Let $C_{ij}$ be the cofactor of $c_{ij}$ in $\Delta$. By Jacobi's Theorem‡ we have

$$\Delta \Delta_{ij} = \Delta_i \Delta_j - C_{ij} \, C_{ji} = C_{ij}^2, \tag{2}$$

since the $\Delta$'s are skew-symmetric and $\Delta_i$ is of odd order.

By another application of Jacobi's Theorem we have

$$\Delta^3 \Delta_{ijkl} = \begin{vmatrix} 0 & C_{ij} & C_{ik} & C_{il} \\ -C_{ij} & 0 & C_{jk} & C_{jl} \\ -C_{ik} & -C_{jk} & 0 & C_{kl} \\ -C_{il} & -C_{jl} & -C_{kl} & 0 \end{vmatrix}$$

(for distinct $i, j, k, l$)

$$= (C_{ij} \, C_{kl} - C_{ik} \, C_{jl} + C_{il} \, C_{jk})^2.$$

* Dénes König, *Theorie der endlichen und unendlichen Graphen*, (Leipzig, 1936), p. 186.

† Cullis, *Matrices and determinoids*, *Vol. II*, (Cambridge, 1918), p. 521.

‡ See e.g. Aitken. *Determinants and matrices*, (Edinburgh, 1939), p. 97.

Hence, by (2),

$$P\,P_{ijkl} = \pm P_{ij}\,P_{kl} \pm P_{ik}\,P_{jl} \pm P_{il}\,P_{jk}. \tag{3}$$

(We need not enquire into the values of the signs.)

Consider a graph $N$, of even order $2m$, whose 0-simplexes are enumerated as $a_1, a_2, \ldots, a_{2m}$. Let $P(N)$ be the Pfaffian derived from (1) by substituting 0 for each $c_{ij}$ for which $a_i$ and $a_j$ are not joined by a 1-simplex. We note that the substitution which changes $P$ into $P(N)$ also changes $P_{rs}$ into $P(N_{rs})$ (apart from sign). From (1) we have the

LEMMA. *A graph $N$ of even order is prime if and only if its Pfaffian $P(N)$ vanishes.*

3. *Prime graphs.* We define a *singularity* of a graph $N$ as a 0-simplex $a_i$ such that, for each $a_j \neq a_i$, $N_{ij}$ is prime.

THEOREM I. *If $N$ is a prime graph of even order, and if $a_r$, $a_s$ are 0-simplexes of $N$ which can be joined in $N$ by a simple arc not having a singularity as an interior point, then $N_{rs}$ is prime.*

First suppose $a_r$, $a_s$ to be joined by a 1-simplex $A_{rs}$. If there were a factor $F$ of $N_{rs}$, then $F \cup A_{rs}$ would be a factor of $N$, contrary to hypothesis.

Next, suppose there are distinct $a_i$, $a_j$, $a_k$, with $a_j$ not a singularity, such that $N_{ij}$ and $N_{jk}$ are prime. Then we can find $a_l$ such that $N_{jl}$ is not prime. Using the lemma we have, by (3),

$$P(N_{ik})\,P(N_{jl}) = 0,$$

where $P(N_{jl}) \neq 0$. Hence $P(N_{ik}) = 0$ and so, by the lemma, $N_{ik}$ is prime.

The theorem follows at once from these two results.

If, in a prime graph $N$, two 0-simplexes $a_r$, $a_s$ are joined by a 1-simplex whenever $N_{rs}$ is prime, we shall say that $N$ is *hyperprime*.

THEOREM II. *If $N$ is a prime graph, we can construct a hyperprime graph $\bar{N}$ which contains $N$ as a subgraph.*

If $N$ is hyperprime, there is nothing to prove. If not, there will be a pair of 0-simplexes $a_r$, $a_s$, not joined by a 1-simplex, such that $N_{rs}$ is prime. Add a new 1-simplex $A_{rs}$ joining them. The resulting graph is prime. For suppose it has a factor $F$. If $A_{rs} \bar{\in} F$, then $F$ is a factor of $N$; if $A_{rs} \in F$, then the intersection of $F$ with $N_{rs}$ is a factor of $N_{rs}$. In either case we have a contradiction.

If the resulting graph is not hyperprime we repeat the process, and so on. Since $N$ is finite the process will eventually terminate in a hyperprime graph of which $N$ is a subgraph.

THEOREM III. *Let $\Sigma$ be the set of singularities of a hyperprime graph $N$ of even order. Then $h_u(\Sigma) > f(\Sigma)$.*

By the definitions of a singularity and a hyperprime graph every pair of 0-simplexes of which one is in $\Sigma$ is joined by a 1-simplex. Further, by Theorem I, every pair of 0-simplexes in the same component of $N_\Sigma$ is joined by a 1-simplex.

If the theorem is false for some $N$ we can, for each component $Q_s$ of odd order of $N_\Sigma$, select a 1-simplex joining a 0-simplex of $Q_s$ to a 0-simplex $D_s$ of $\Sigma$; and we can arrange that all the $D_s$ are distinct. For $\Sigma$ and every component of $N_\Sigma$ we can then select other 1-simplexes joining up the remaining even number of 0-simplexes in pairs. We thus obtain a factor of $N$, contrary to its definition.

THEOREM IV. *A graph $N$ is prime if and only if there is a subset $S$ of its 0-simplexes such that $h_u(S) > f(S)$.*

The case in which the order of $N$ is odd is trivial. (Take the null set as $S$.)

Suppose $N$ is of even order and that, for some $S$, $h_u(S) > f(S)$. Any factor $F$ of $N$ must evidently contain a 1-simplex joining a 0-simplex of a given component of $N_S$ of odd order to a 0-simplex not in that component and therefore in $S$. Hence there must be more 1-simplexes of $F$ incident with members of $S$ than there are members of $S$, which is absurd since $F$ is regular and of the first degree. Consequently $N$ is prime.

Next, suppose $N$ prime. Then we can construct a hyperprime graph $\bar{N}$ of which $N$ is a subgraph (Theorem II). Let $\Sigma$ be the set of singularities of $\bar{N}$. Then $h_u(\Sigma) > f(\Sigma)$ is an inequality true for $\bar{N}$ (Theorem III). Hence it is true also for $N$, for each component of odd order of $\bar{N}_\Sigma$ must contain at least one component of odd order of $N_\Sigma$.

In virtue of the lemma it is easily seen that this theorem is equivalent to the following proposition.

*Let $M$ be a skew-symmetric matrix in which, of the elements above the diagonal, some are zero and the others independent indeterminates. Then a necessary and sufficient condition for $|M|$ to vanish is that $M$ shall contain a diagonal submatrix $M_0$ which is a direct product of skew-symmetric matrices of which the number having odd order exceeds the difference of the orders of $M$ and $M_0$.*

4. *An existence theorem.*

THEOREM V. *Let $N$ be a connected graph of even order which is regular and of degree $\sigma$. Suppose further that $N$ has no isthmoid whose rank is less than $\sigma-1$. Then at least one factor of $N$ exists.*

Let $S$ be any isthmoid of $N$, and let $C$ be any component of $N_S$. Let $L(C)$ be the number of 1-simplexes having one end in $C$ and the other in $S$. If the order $n(C)$ of $C$ is odd, we have $L(C) \geqslant \sigma$. For, since no isthmoid has rank less than $\sigma-1$, the only other possibility is $L(C) = \sigma-1$. In that case the number of 1-simplexes contained in $C$ would be $\frac{1}{2}[\sigma n(C)-\sigma+1]$ which is not an integer. So, if $k$ is the number of 1-simplexes having one end in $S$ and the other in $N_S$, we have

$$\sigma h_u(S) \leqslant k \leqslant \sigma f(S). \tag{4}$$

Thus for no $S$ does $h_u(S)$ exceed $f(S)$, and so, by Theorem IV, $N$ has a factor.

COROLLARY. *Let $A$ be any 1-simplex of $N$. Then $N$ has a factor which contains $A$.*

Let the vertices of $A$ be $a_r$ and $a_s$.

Suppose that the corollary is false for some $N$. Then $N_{rs}$ is prime. So, by Theorem IV, there is an isthmoid $S$ of $N_{rs}$ such that $h_u(S) > f(S)$ in $N_{rs}$.

Let $S'$ be the set formed by adding $a_r$ and $a_s$ to $S$. Hereafter functions of $S$ will refer to $N_{rs}$, functions of $S'$ to $N$. Clearly

$$f(S') = f(S)+2 \tag{5}$$

and

$$h_u(S') = h_u(S) \tag{6}$$

[for $(N_{rs})_S$ is the same as $N_{S'}$]. Referring to the proof of the main theorem, we see that $f(S') > h_u(S')$; for the second equality in (4) applies only if each 1-simplex incident with a member of $S'$ is also incident with a 0-simplex of $N_{S'}$. This is not true of $A$. But since $N$ is of even order the numbers $f(S')$ and $h_u(S')$ must have the same parity. Hence

$$f(S') \geqslant h_u(S')+2,$$

$$f(S) \geqslant h_u(S) \qquad \text{[by (5) and (6)]}.$$

This is contrary to the definition of $S$.

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